

GRAPH DIRECTED COALESCENCE HIDDEN VARIABLE FRACTAL INTERPOLATION FUNCTIONS

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ABSTRACT. Fractal interpolation function (FIF) is a special type of continuous function which interpolates certain data set and the attractor of the Iterated function system (IFS) corresponding to the data set is the graph of the FIF. Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) is both self-affine and non self-affine in nature depending on the free variables and constrained free variables for a generalized IFS. In this article graph directed iterated function system for a finite number of generalized data sets is considered and it is shown that the projections of the attractors on \mathbb{R}^2 is the graph of the CHFIFs interpolating the corresponding data sets.

1. INTRODUCTION

The concept of fractal interpolation function (FIF) based on an iterated function system (IFS) as a fixed point of Hutchinson's operator is introduced by Barnsley [?, ?]. The attractor of the IFS is the graph of the fractal function interpolating certain data set. These FIFs are generally self-affine in nature. The idea has been extended to a generalized data set in \mathbb{R}^3 such that the projection of the graph of the corresponding FIF onto \mathbb{R}^2 provides a non self-affine interpolation function namely Hidden variable FIFs for a given data set $\{(x_n, y_n) : n = 0, 1, \dots, N\}$ [?]. Chand and Kapoor [?], introduced the concept of Coalescence hidden variable FIFs which are both self-affine and non self-affine for generalized IFS. The extra degree of freedom is useful to adjust the shape and fractal dimension of the interpolation functions. In [?], Barnsley et al. proved existence of a differentiable FIF. The continuous but nowhere differentiable fractal function namely α -fractal interpolation function f^α is introduced by Navascues as perturbation of a continuous function f on a compact interval I of \mathbb{R} [?, ?]. Interested reader can see for the theory and application of α -fractal interpolation function f^α which has been extensively explored by Navascues [?, ?, ?].

In [?] Deniz et al. considered graph-directed iterated function system for finite number of data sets and proved the existence of fractal functions interpolating corresponding data sets with graphs as the attractor of the GDIFS.

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In the present work, generalized GDIFS for generalized interpolation data sets in \mathbb{R}^3 has taken. It is shown that, corresponding to the data sets there exists CHFIFs whose graph is the projection on \mathbb{R}^2 of the attractors of the GDIFS.

2. PRELIMINARIES

2.1. Iterated Function System. Let $\mathcal{X} \subset \mathbb{R}^n$ and $(\mathcal{X}, d_{\mathcal{X}})$ be a complete metric space. Also assume $\mathcal{H}(\mathcal{X}) = \{S \subset \mathcal{X}; S \neq \emptyset, S \text{ is compact in } \mathcal{X}\}$ with the Hausdorff metric $d_{\mathcal{H}}(A, B)$ defined as $d_{\mathcal{H}}(A, B) = \max\{d_{\mathcal{X}}(A, B), d_{\mathcal{X}}(B, A)\}$, where $d_{\mathcal{X}}(A, B) = \max_{x \in A} \min_{y \in B} d_{\mathcal{X}}(x, y)$ for any two sets A, B in $\mathcal{H}(\mathcal{X})$. $(\mathcal{H}, d_{\mathcal{H}})$ is a complete metric space whenever $(\mathcal{X}, d_{\mathcal{X}})$ is complete. Let for $i = 1, 2, \dots, N$, $w_i : \mathcal{X} \rightarrow \mathcal{X}$ are continuous maps then $\{\mathcal{X}; w_i : i = 1, 2, \dots, N\}$ is called an iterated function system (IFS). If the maps w_i 's are contraction then, the set valued Hutchinson operator $W : \mathcal{H}(\mathcal{X}) \rightarrow \mathcal{H}(\mathcal{X})$ defined by $W(B) = \bigcup_{i=1}^N w_i(B)$, where $w_i(B) := \{w_i(b) : b \in B\}$ is also contraction. Then by Banach fixed point theorem, there exists a unique set $A \in \mathcal{H}(\mathcal{X})$ such that $A = W(A) = \bigcup_{i=1}^N w_i(A)$. The set A is called the attractor associated with the IFS $\{\mathcal{X}; w_i : i = 1, 2, \dots, N\}$.

2.2. Fractal Interpolation Function. Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \dots, N\}$ be given, where $\Delta : x_0 < x_1 < \dots < x_N$ is a partition of the closed interval $I = [x_0, x_N]$ and $y_i \in [g_1, g_2] \subset \mathbb{R}$, $i = 0, 1, \dots, N$. Set $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, \dots, N$ and $K = I \times [g_1, g_2]$. Let $L_i : I \rightarrow I_i, i = 1, 2, \dots, N$, be contraction homeomorphisms such that

$$(1) \quad L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i,$$

$$(2) \quad |L_i(c_1) - L_i(c_2)| \leq d|c_1 - c_2| \text{ for all } c_1 \text{ and } c_2 \text{ in } I,$$

for some $0 \leq d < 1$. Furthermore, let $H_i : K \rightarrow \mathbb{R}, i = 1, 2, \dots, N$, be given continuous functions such that

$$(3) \quad H_i(x_0, y_0) = y_{i-1}, \quad H_i(x_N, y_N) = y_i,$$

$$(4) \quad |H_i(x, \xi_1) - H_i(x, \xi_2)| \leq |\alpha_i| |\xi_1 - \xi_2|$$

for all $x \in I$ and for all ξ_1 and ξ_2 in $[g_1, g_2]$, for some $\alpha_i \in (-1, 1), i = 1, 2, \dots, N$. Define mappings $W_i : K \rightarrow I_i \times \mathbb{R}, i = 1, 2, \dots, N$ by

$$W_i(x, y) = (L_i(x), H_i(x, y)) \text{ for all } (x, y) \in K.$$

Then

$$(5) \quad \{K; W_i(x, y) : i = 1, 2, \dots, N\}$$

constitutes an IFS. Barnsley [?] proved that the IFS $\{K; W_i : i = 1, 2, \dots, N\}$ defined above has a unique attractor G where G is the graph of a continuous function $f : I \rightarrow \mathbb{R}$ which obeys $f(x_i) = y_i$ for $i = 0, 1, \dots, N$. This function f is called a fractal interpolation function (FIF) or simply fractal function and it is the unique function satisfying the following fixed point equation

$$(6) \quad f(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x))) \text{ for all } x \in I_i, i = 1, 2, \dots, N.$$

The widely studied FIFs so far are defined by the iterated mappings

$$(7) \quad L_i(x) = a_i x + d_i, \quad F_i(x, y) = \alpha_i y + q_i(x), \quad i = 1, 2, \dots, N,$$

where the real constants a_i and d_i are determined by the condition (1) as

$$(8) \quad a_i = \frac{(x_i - x_{i-1})}{(x_N - x_0)} \quad \text{and} \quad d_i = \frac{(x_N x_{i-1} - x_0 x_i)}{(x_N - x_0)},$$

and $q_i(x)$'s are suitable continuous functions such that the conditions (3) and (4) hold. For each i , α_i is a free parameter with $|\alpha_i| < 1$ and is called a vertical scaling factor of the transformation W_i . Then the vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is called the scale vector of the IFS. If $q_i(x)$ is taken as linear then the corresponding FIF is known as affine FIF (AFIF).

2.3. Coalescence FIF. To construct a Coalescence Hidden-variable Fractal Interpolation Functions, a set of real parameters z_i for $i = 1, 2, \dots, N$ are introduced and the generalized interpolation data $\{(x_i, y_i, z_i) \in \mathbb{R}^3 : i = 0, 1, \dots, N\}$ is considered. Then define the maps $w_i : I \times \mathbb{R}^2 \rightarrow I_i \times \mathbb{R}^2, i = 1, 2, \dots, N$ by

$$w_i(x, y, z) = (L_i(x), F_i(x, y, z))$$

where, $L_i : I \rightarrow I_i, i = 1, 2, \dots, N$ are given in (7), and the functions $F_i : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F_i(x, y, z) = (F_i^1(x, y, z), F_i^2(x, y, z)) = (\alpha_i y + \beta_i z + c_i x + d_i, \gamma_i z + e_i x + f_i)$ satisfy the join-up conditions

$$F_i(x_0, y_0, z_0) = (y_{i-1}, z_{i-1}) \quad \text{and} \quad F_i(x_N, y_N, z_N) = (y_i, z_i).$$

Here α_i, γ_i are free variables with $|\alpha_i| < 1, |\gamma_i| < 1$ and β_i are constrained variable such that $|\beta_i| + |\gamma_i| < 1$. Then the generalized IFS

$$\{I \times \mathbb{R}^2; w_i(x, y, z) : i = 1, 2, \dots, N\}$$

has an attractor G such that $G = \bigcup_{i=1}^N w_i(G) = \bigcup_{i=1}^N \{w_i(x, y, z) : (x, y, z) \in G\}$ [?]. The attractor G is the graph of a vector valued function $f : I \rightarrow \mathbb{R}^2$ such that $f(x_i) = (y_i, z_i)$ for $i = 0, 1, \dots, N$ and $G = \{(x, f(x)) : x \in I, f(x) = (y(x), z(x))\}$. If $f = (f_1, f_2)$, then the projection of the attractor G on \mathbb{R}^2 is the graph of the function f_1 which satisfies $f_1(x_i) = y_i$ and is of the form

$$f_1(L_i(x)) = F_i^1(x, f_1(x), f_2(x)) = \alpha_i f_1(x) + \beta_i f_2(x) + c_i x + d_i, \quad x \in I$$

is known as CHFIF corresponding to the data $\{(x_i, y_i) \in I \times \mathbb{R} : i = 0, 1, \dots, N\}$.

2.4. Graph-directed Iterated Function Systems. Let $G = (V, E)$ be a directed graph where V denote the set of vertices and E is the set of edges. For all $u, v \in V$, let E^{uv} denote the set of edges from u to v with elements $e_i^{uv}, i = 1, 2, \dots, K^{uv}$ where K^{uv} denotes the number of elements of E^{uv} . An iterated function system realizing the graph G is given by a collection of metric spaces $(X^v, \rho^v), v \in V$, and of contraction mappings $w_i^{uv} : X^v \rightarrow X^u$ corresponding to the edge e_i^{uv} in the opposite direction of e_i^{uv} . An attractor (or invariant

list) for such an iterated function system is a list of nonempty compact sets $A^u \subset X^u$ such that for all $u \in V$,

$$A^u = \bigcup_{v \in V} \bigcup_{i=1}^{K^{uv}} w_i^{uv}(A^v).$$

Then $(X^u; w_i^{uv})$ is the graph directed iterated function system (GDIFS) realizing the graph G [?, ?].

Example 2.1. *One can see [?, ?].*

3. GRAPH DIRECTED COALESCENCE FIF

In this section, for a finite number of data sets, generalized graph-directed iterated function system (GDIFS) is defined so that projection of each attractor on \mathbb{R}^2 is the graph of a CHFIF which interpolates the corresponding data set and call it as graph-directed coalescence hidden-variable fractal interpolation function. For simplicity, only two sets of data are considered. Let the two data sets as

$$\begin{aligned} D^1 &= \{(x_0^1, y_0^1), (x_1^1, y_1^1), \dots, (x_N^1, y_N^1)\} \\ D^2 &= \{(x_0^2, y_0^2), (x_1^2, y_1^2), \dots, (x_M^2, y_M^2)\} \end{aligned}$$

with $N, M \geq 2$ and

$$(9) \quad \frac{x_i^1 - x_{i-1}^1}{x_M^2 - x_0^2} < 1 \text{ and } \frac{x_j^2 - x_{j-1}^2}{x_N^1 - x_0^1} < 1$$

for all $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$. By introducing two set of real parameters z_i^1, z_j^2 for $i = 1, 2, \dots, N$ and $j = 1, 2, \dots, M$, consider the two generalized data set as

$$\begin{aligned} \mathcal{D}^1 &= \{(x_0^1, y_0^1, z_0^1), (x_1^1, y_1^1, z_1^1), \dots, (x_N^1, y_N^1, z_N^1)\} \\ \mathcal{D}^2 &= \{(x_0^2, y_0^2, z_0^2), (x_1^2, y_1^2, z_1^2), \dots, (x_M^2, y_M^2, z_M^2)\}. \end{aligned}$$

Now consider the directed graph $G = (V, E)$ with $V = \{1, 2\}$ is such that

$$K^{11} + K^{12} = N \text{ and } K^{21} + K^{22} = M.$$

give a picture.

To construct a generalized GDIFS associated with the data \mathcal{D}^r , ($r = 1, 2$) and realizing the graph G consider the functions $w_n^{rs} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as

$$w_n^{rs}(x, y, z) = (L_n^{rs}(x), F_n^{rs}(x, y, z)), \quad n = 1, 2, \dots, K^{rs}$$

are such that

$$\begin{aligned} \bullet & \begin{cases} w_n^{11}(x_0^1, y_0^1, z_0^1) = (x_{n-1}^1, y_{n-1}^1, z_{n-1}^1) \\ w_n^{11}(x_N^1, y_N^1, z_N^1) = (x_n^1, y_n^1, z_n^1) \end{cases} \quad \text{for } n = 1, 2, \dots, K^{11} \\ \bullet & \begin{cases} w_{n-K^{11}}^{12}(x_0^2, y_0^2, z_0^2) = (x_{n-1}^1, y_{n-1}^1, z_{n-1}^1) \\ w_{n-K^{11}}^{11}(x_M^2, y_M^2, z_M^2) = (x_n^1, y_n^1, z_n^1) \end{cases} \quad \text{for } n = K^{11} + 1, \dots, K^{11} + K^{12} = N \end{aligned}$$

- $\begin{cases} w_n^{21}(x_0^1, y_0^1, z_0^1) = (x_{n-1}^2, y_{n-1}^2, z_{n-1}^2) \\ w_n^{21}(x_N^1, y_N^1, z_N^1) = (x_n^2, y_n^2, z_n^2) \end{cases} \quad \text{for } n = 1, 2, \dots, K^{21}$
- $\begin{cases} w_{n-K^{21}}^{22}(x_0^2, y_0^2, z_0^2) = (x_{n-1}^2, y_{n-1}^2, z_{n-1}^2) \\ w_{n-K^{21}}^{22}(x_M^2, y_M^2, z_M^2) = (x_n^2, y_n^2, z_n^2) \end{cases} \quad \text{for } n = K^{21} + 1, \dots, K^{21} + K^{22} = M$

From each of the above conditions, the following can derive respectively.

$$(10) \quad \begin{cases} a_n^{11}x_0^1 + b_n^{11} = x_{n-1}^1 \\ a_n^{11}x_N^1 + b_n^{11} = x_n^1 \\ c_n^{11}x_0^1 + \alpha_n^{11}y_0^1 + \beta_n^{11}z_0^1 + d_n^{11} = y_{n-1}^1 \\ c_n^{11}x_N^1 + \alpha_n^{11}y_N^1 + \beta_n^{11}z_N^1 + d_n^{11} = y_n^1 \\ e_n^{11}x_0^1 + \gamma_n^{11}z_0^1 + f_n^{11} = z_{n-1}^1 \\ e_n^{11}x_N^1 + \gamma_n^{11}z_N^1 + f_n^{11} = z_n^1 \end{cases} \quad \text{for } n = 1, 2, \dots, K^{11}$$

$$(11) \quad \begin{cases} a_{n-K^{11}}^{12}x_0^2 + b_{n-K^{11}}^{12} = x_{n-1}^1 \\ a_{n-K^{11}}^{12}x_M^2 + b_{n-K^{11}}^{12} = x_n^1 \\ c_{n-K^{11}}^{12}x_0^2 + \alpha_{n-K^{11}}^{12}y_0^2 + \beta_{n-K^{11}}^{12}z_0^2 + d_{n-K^{11}}^{12} = y_{n-1}^1 \\ c_{n-K^{11}}^{12}x_M^2 + \alpha_{n-K^{11}}^{12}y_M^2 + \beta_{n-K^{11}}^{12}z_M^2 + d_{n-K^{11}}^{12} = y_n^1 \\ e_{n-K^{11}}^{12}x_0^2 + \gamma_{n-K^{11}}^{12}z_0^2 + f_{n-K^{11}}^{12} = z_{n-1}^1 \\ e_{n-K^{11}}^{12}x_M^2 + \gamma_{n-K^{11}}^{12}z_M^2 + f_{n-K^{11}}^{12} = z_n^1 \end{cases} \quad \text{for } n = K^{11} + 1, \dots, N$$

$$(12) \quad \begin{cases} a_n^{21}x_0^1 + b_n^{21} = x_{n-1}^2 \\ a_n^{21}x_N^1 + b_n^{21} = x_n^2 \\ c_n^{21}x_0^1 + \alpha_n^{21}y_0^1 + \beta_n^{21}z_0^1 + d_n^{21} = y_{n-1}^2 \\ c_n^{21}x_N^1 + \alpha_n^{21}y_N^1 + \beta_n^{21}z_N^1 + d_n^{21} = y_n^2 \\ e_n^{21}x_0^1 + \gamma_n^{21}z_0^1 + f_n^{21} = z_{n-1}^2 \\ e_n^{21}x_N^1 + \gamma_n^{21}z_N^1 + f_n^{21} = z_n^2 \end{cases} \quad \text{for } n = 1, 2, \dots, K^{21}$$

$$(13) \quad \begin{cases} a_{n-K^{21}}^{22}x_0^2 + b_{n-K^{21}}^{22} = x_{n-1}^2 \\ a_{n-K^{21}}^{22}x_M^2 + b_{n-K^{21}}^{22} = x_n^2 \\ c_{n-K^{21}}^{22}x_0^2 + \alpha_{n-K^{21}}^{22}y_0^2 + \beta_{n-K^{21}}^{22}z_0^2 + d_{n-K^{21}}^{22} = y_{n-1}^2 \\ c_{n-K^{21}}^{22}x_M^2 + \alpha_{n-K^{21}}^{22}y_M^2 + \beta_{n-K^{21}}^{22}z_M^2 + d_{n-K^{21}}^{22} = y_n^2 \\ e_{n-K^{21}}^{22}x_0^2 + \gamma_{n-K^{21}}^{22}z_0^2 + f_{n-K^{21}}^{22} = z_{n-1}^2 \\ e_{n-K^{21}}^{22}x_M^2 + \gamma_{n-K^{21}}^{22}z_M^2 + f_{n-K^{21}}^{22} = z_n^2 \end{cases} \quad \text{for } n = K^{21} + 1, \dots, M$$

From the linear system of equations (10), (11), (12) and (13) the constants a_i^{rs} , b_i^{rs} , c_i^{rs} , d_i^{rs} , e_i^{rs} and f_i^{rs} for $r, s \in \{1, 2\}$, $i = 1, 2, \dots, K^{rs}$ are determined as follows

$$\begin{aligned}
a_n^{11} &= \frac{x_n^1 - x_{n-1}^1}{x_N^1 - x_0^1} & a_n^{12} &= \frac{x_n^1 - x_{n-1}^1}{x_M^2 - x_0^2} \\
b_n^{11} &= \frac{x_N^1 x_{n-1}^1 - x_0^1 x_n^1}{x_N^1 - x_0^1} & b_n^{12} &= \frac{x_M^2 x_{n-1}^1 - x_0^2 x_n^1}{x_M^2 - x_0^2} \\
c_n^{11} &= \frac{y_n^1 - y_{n-1}^1 - \alpha_n^{11}(y_N^1 - y_0^1) - \beta_n^{11}(z_N^1 - z_0^1)}{x_N^1 - x_0^1} & c_n^{12} &= \frac{y_n^1 - y_{n-1}^1 - \alpha_n^{12}(y_M^2 - y_0^2) - \beta_n^{12}(z_M^2 - z_0^2)}{x_M^2 - x_0^2} \\
d_n^{11} &= \frac{x_N^1 y_{n-1}^1 - x_0^1 y_n^1 - \alpha_n^{11}(x_N^1 y_0^1 - x_0^1 y_N^1) - \beta_n^{11}(x_N^1 z_0^1 - x_0^1 z_N^1)}{x_N^1 - x_0^1} & d_n^{12} &= \frac{x_M^2 y_{n-1}^1 - x_0^2 y_n^1 - \alpha_n^{12}(x_M^2 y_0^2 - x_0^2 y_M^2) - \beta_n^{12}(x_M^2 z_0^2 - x_0^2 z_M^2)}{x_M^2 - x_0^2} \\
e_n^{11} &= \frac{z_n^1 - z_{n-1}^1 - \gamma_n^{11}(z_N^1 - z_0^1)}{x_N^1 - x_0^1} & e_n^{12} &= \frac{z_n^1 - z_{n-1}^1 - \gamma_n^{12}(z_M^2 - z_0^2)}{x_M^2 - x_0^2} \\
f_n^{11} &= \frac{x_N^1 z_{n-1}^1 - x_0^1 z_n^1 - \gamma_n^{11}(x_N^1 z_0^1 - x_0^1 z_N^1)}{x_N^1 - x_0^1} & f_n^{12} &= \frac{x_M^2 z_{n-1}^1 - x_0^2 z_n^1 - \gamma_n^{12}(x_M^2 z_0^2 - x_0^2 z_M^2)}{x_M^2 - x_0^2} \\
a_n^{21} &= \frac{x_n^2 - x_{n-1}^2}{x_N^1 - x_0^1} & a_n^{22} &= \frac{x_n^2 - x_{n-1}^2}{x_M^2 - x_0^2} \\
b_n^{21} &= \frac{x_N^1 x_{n-1}^2 - x_0^1 x_n^2}{x_N^1 - x_0^1} & b_n^{22} &= \frac{x_M^2 x_{n-1}^2 - x_0^2 x_n^2}{x_M^2 - x_0^2} \\
c_n^{21} &= \frac{y_n^2 - y_{n-1}^2 - \alpha_n^{21}(y_N^1 - y_0^1) - \beta_n^{21}(z_N^1 - z_0^1)}{x_N^1 - x_0^1} & c_n^{22} &= \frac{y_n^2 - y_{n-1}^2 - \alpha_n^{22}(y_M^2 - y_0^2) - \beta_n^{22}(z_M^2 - z_0^2)}{x_M^2 - x_0^2} \\
d_n^{21} &= \frac{x_N^1 y_{n-1}^2 - x_0^1 y_n^2 - \alpha_n^{21}(x_N^1 y_0^1 - x_0^1 y_N^1) - \beta_n^{21}(x_N^1 z_0^1 - x_0^1 z_N^1)}{x_N^1 - x_0^1} & d_n^{22} &= \frac{x_M^2 y_{n-1}^2 - x_0^2 y_n^2 - \alpha_n^{22}(x_M^2 y_0^2 - x_0^2 y_M^2) - \beta_n^{22}(x_M^2 z_0^2 - x_0^2 z_M^2)}{x_M^2 - x_0^2} \\
e_n^{21} &= \frac{z_n^2 - z_{n-1}^2 - \gamma_n^{21}(z_N^1 - z_0^1)}{x_N^1 - x_0^1} & e_n^{22} &= \frac{z_n^2 - z_{n-1}^2 - \gamma_n^{22}(z_M^2 - z_0^2)}{x_M^2 - x_0^2} \\
f_n^{21} &= \frac{x_N^1 z_{n-1}^2 - x_0^1 z_n^2 - \gamma_n^{21}(x_N^1 z_0^1 - x_0^1 z_N^1)}{x_N^1 - x_0^1} & f_n^{22} &= \frac{x_M^2 z_{n-1}^2 - x_0^2 z_n^2 - \gamma_n^{22}(x_M^2 z_0^2 - x_0^2 z_M^2)}{x_M^2 - x_0^2}
\end{aligned}$$

The following theorem shows that each maps w_n^{rs} is contraction with respect to metric equivalent to the Euclidean metric and ensures the existence of attractors of generalized GDIFS.

Theorem 3.1. *Let $\{\mathbb{R}^3; w_n^{rs}, n = 1, 2, \dots, K^{rs}\}$ be the generalized GDIFS defined above realizing the graph and associated with the data sets $\mathcal{D}^r, (r = 1, 2)$ which satisfy (9). If $|\alpha_n^{rs}| < 1, |\gamma_n^{rs}| < 1$ and β_n^{rs} is chosen such that $|\beta_n^{rs}| + |\gamma_n^{rs}| < 1$ for all $r, s \in \{1, 2\}$ and $n = 1, 2, \dots, K^{rs}$. Then there exists a metric δ on \mathbb{R}^3 equivalent to the Euclidean metric, such that the GDIFS is hyperbolic with respect to δ . In particular, there exists non empty compact sets G^r such that*

$$G^r = \bigcup_{s=1}^2 \bigcup_{n=1}^{K^{rs}} w_n^{rs}(G^s).$$

Proof. Proof follows in the similar line of Theorem 2.1.1, [?] and using above condition (9). \square

Following is the main result regarding existence of coalescence Hidden-variable FIFs for generalized GDIFS.

Theorem 3.2. *Let $G^r, r \in V$ be the attractors of the generalized GDIFS as in Theorem 3.1. Then $G^r, r \in V$ is the graph of a vector valued continuous function $f^r : I^r \rightarrow \mathbb{R}^2$ such that for $r \in V, f^r(x_n^r) = (y_n^r, z_n^r)$ for all $n = 1, 2, \dots, N^r$. If $f^r = (f_1^r, f_2^r)$ then the projection of the attractors $G^r, r \in V$ on \mathbb{R}^2 is the graph of the continuous function $f_1^r : I^r \rightarrow \mathbb{R}$ known as CHFIF such that for $r \in V, f^r(x_n^r) = (y_n^r)$. That is $G^r|_{\mathbb{R}^2} = \{(x, f_1^r(x)) : x \in I^r\}$*

Proof. Consider the vector valued function spaces

$$\mathcal{F} = \{f : [x_0^1, x_N^1] \rightarrow \mathbb{R}^2 \text{ continuous such that } f(x_0^1) = (y_0^1, z_0^1), f(x_N^1) = (y_N^1, z_N^1)\}$$

$$\mathcal{H} = \{h : [x_0^2, x_M^2] \rightarrow \mathbb{R}^2 \text{ continuous such that } h(x_0^2) = (y_0^2, z_0^2), h(x_M^2) = (y_M^2, z_M^2)\}$$

with metrics

$$d_{\mathcal{F}}(f_1, f_2) = \sup_{x \in [x_0^1, x_N^1]} \|f_1(x) - f_2(x)\|$$

$$d_{\mathcal{H}}(h_1, h_2) = \sup_{x \in [x_0^2, x_M^2]} \|h_1(x) - h_2(x)\|$$

respectively, where $\|\cdot\|$ denotes a norm on \mathbb{R}^2 . Since $(\mathcal{F}, d_{\mathcal{F}})$ and $(\mathcal{H}, d_{\mathcal{H}})$ are complete metric spaces, then $(\mathcal{F} \times \mathcal{H}, d)$ is also a complete metric space where

$$d((f_1, h_1), (f_2, h_2)) = \max\{d_{\mathcal{F}}(f_1, f_2), d_{\mathcal{H}}(h_1, h_2)\}.$$

Following are the affine maps.

$$I : [x_0^1, x_N^1] \rightarrow [x_{n-1}^1, x_n^1], I_n^1(x) = a_n^{11}x + b_n^{11} \text{ for } n = 1, 2, \dots, K^{11}$$

$$I : [x_0^2, x_M^2] \rightarrow [x_{n-K^{11}}^1, x_n^1], I_n^1(x) = a_{n-K^{11}}^{12}x + b_{n-K^{11}}^{12} \text{ for } n = K^{11} + 1, \dots, K^{12}$$

$$J : [x_0^1, x_N^1] \rightarrow [x_{n-1}^2, x_n^2], I_n^2(x) = a_n^{21}x + b_n^{21} \text{ for } n = 1, 2, \dots, K^{21}$$

$$J : [x_0^2, x_M^2] \rightarrow [x_{n-K^{21}}^2, x_n^2], I_n^2(x) = a_{n-K^{21}}^{22}x + b_{n-K^{21}}^{22} \text{ for } n = K^{21} + 1, \dots, K^{22}$$

Now define the mapping

$$T : \mathcal{F} \times \mathcal{H} \rightarrow \mathcal{F} \times \mathcal{H}$$

$$T(f, h)(x, y) = (\tilde{f}(x), \tilde{h}(y))$$

where for $x \in [x_{n-1}^1, x_n^1]$

(14)

$$\tilde{f}(x) = \begin{cases} (c_n^{11} I_n^{-1}(x) + \alpha_n^{11} y_f^1(I_n^{-1}(x)) + \beta_n^{11} z_f^1(I_n^{-1}(x)) + d_n^{11}, & \text{for } n = 1, 2, \dots, K^{11} \\ \gamma_n^{11} z_f^1(I_n^{-1}(x)) + e_n^{11} I_n^{-1}(x) + f_n^{11} & \\ (c_{n-K^{11}}^{12} I_n^{-1}(x) + \alpha_{n-K^{11}}^{12} y_h^2(I_n^{-1}(x)) + \beta_{n-K^{11}}^{12} z_h^2(I_n^{-1}(x)) + d_{n-K^{11}}^{12}, & \\ \gamma_{n-K^{11}}^{12} z_h^2(I_n^{-1}(x)) + e_{n-K^{11}}^{12} I_n^{-1}(x) + f_{n-K^{11}}^{12} & \text{for } n = K^{11} + 1, \dots, N. \end{cases}$$

and for $x \in [x_{m-1}^2, x_m^2]$

(15)

$$\tilde{h}(x) = \begin{cases} (c_m^{21} J_m^{-1}(x) + \alpha_m^{21} y_f^1(J_m^{-1}(x)) + \beta_m^{21} z_f^1(J_m^{-1}(x)) + d_m^{21}, & \text{for } m = 1, 2, \dots, K^{21} \\ \gamma_m^{21} z_f^1(J_m^{-1}(x)) + e_m^{21} J_m^{-1}(x) + f_m^{21} & \\ (c_{m-K^{21}}^{22} J_m^{-1}(x) + \alpha_{m-K^{21}}^{22} y_h^2(J_m^{-1}(x)) + \beta_{m-K^{21}}^{22} z_h^2(J_m^{-1}(x)) + d_{m-K^{21}}^{22}, & \\ \gamma_{m-K^{21}}^{22} z_h^2(J_m^{-1}(x)) + e_{m-K^{21}}^{22} J_m^{-1}(x) + f_{m-K^{21}}^{22} & \text{for } m = K^{21} + 1, \dots, M. \end{cases}$$

Now using equations (10) – (13) it is clear that,

$$\tilde{f}(x_0^1) = F_1(I_n^{-1}(x), y_f^1(I_n^{-1}(x)), z_f^1(I_n^{-1}(x))) = (y_0^1, z_0^1)$$

$$\tilde{f}(x_N^1) = F_N(I_n^{-1}(x), y_h^2(I_n^{-1}(x)), z_h^2(I_n^{-1}(x))) = (y_N^1, z_N^1).$$

Similarly, $\tilde{h}(x_0^2) = (y_0^2, z_0^2)$, $\tilde{h}(x_M^2) = (y_M^2, z_M^2)$. Which proves that T maps $\mathcal{F} \times \mathcal{H}$ into itself. Since for each $n = 1, 2, \dots, N$, $I_n^{-1}(x)$ is continuous and therefore, \tilde{f} is continuous on each subintervals $[x_{n-1}^1, x_n^1]$.

For $n = 1, 2, \dots, K^{11}$, using (10) it follows that $\tilde{f}(x_n^{1-}) = \tilde{f}(x_n^{1+}) = (y_n^1, z_n^1)$.

For $n = K^{11} + 1, \dots, N - 1$, using (11) it follows that $\tilde{f}(x_n^{1-}) = \tilde{f}(x_n^{1+}) = (y_n^1, z_n^1)$.

For $n = K^{11}$, using (10) and (11) it follows that $\tilde{f}(x_n^{1-}) = \tilde{f}(x_n^{1+}) = (y_n^1, z_n^1)$ since $I_n^{-1}(x_n^1) = x_N^1$ and $I_{n+1}^{-1}(x_n^1) = x_0^2$.

Hence \tilde{f} is continuous on I . Similarly it can be shown that \tilde{h} is continuous on J . Consequently T is continuous.

To show that T is a contraction map on $\mathcal{F} \times \mathcal{H}$, let $T(f_1, f_2) = (\tilde{f}_1, \tilde{f}_2)$ and $T(h_1, h_2) = (\tilde{h}_1, \tilde{h}_2)$. Now

$$\begin{aligned} \sup_{x \in [x_0^1, x_{K^{11}}^1]} \{\|\tilde{f}_1(x) - \tilde{f}_2(x)\|\} &= \max_{\substack{n=1,2,\dots,K^{11} \\ x \in [x_{n-1}^1, x_n^1]}} \{\|\alpha_n^{11}(y_{f_1}^1(I_n^{-1}(x)) - y_{f_2}^1(I_n^{-1}(x))) \\ &\quad + \beta_n^{11}(z_{f_1}^1(I_n^{-1}(x)) - z_{f_2}^1(I_n^{-1}(x))), \gamma_n^{11}(z_{f_1}^1(I_n^{-1}(x)) - z_{f_2}^1(I_n^{-1}(x)))\|\} \\ &\leq \delta^{11} \max_{\substack{n=1,2,\dots,K^{11} \\ x \in [x_{n-1}^1, x_n^1]}} \{y_{f_1}^1(I_n^{-1}(x)) - y_{f_2}^1(I_n^{-1}(x)) + z_{f_1}^1(I_n^{-1}(x)) - z_{f_2}^1(I_n^{-1}(x)), \\ &\quad z_{f_1}^1(I_n^{-1}(x)) - z_{f_2}^1(I_n^{-1}(x))\} \\ &\leq \delta^{11} d_{\mathcal{F}}(f_1, f_2). \end{aligned}$$

$$\begin{aligned} \sup_{x \in [x_{K^{11}}^1, x_N^1]} \{\|\tilde{f}_1(x) - \tilde{f}_2(x)\|\} &= \max_{\substack{n=K^{11}+1,\dots,N \\ x \in [x_{n-1}^1, x_n^1]}} \{\|\alpha_{n-K^{11}}^{12}(y_{h_1}^2(I_n^{-1}(x)) - y_{h_2}^2(I_n^{-1}(x))) \\ &\quad + \beta_{n-K^{11}}^{12}(z_{h_1}^2(I_n^{-1}(x)) - z_{h_2}^2(I_n^{-1}(x))), \gamma_{n-K^{11}}^{12}(z_{h_1}^2(I_n^{-1}(x)) - z_{h_2}^2(I_n^{-1}(x)))\|\} \\ &\leq \delta^{12} \max_{\substack{n=K^{11}+1,\dots,N \\ x \in [x_{n-1}^1, x_n^1]}} \{y_{h_1}^2(I_n^{-1}(x)) - y_{h_2}^2(I_n^{-1}(x)) + z_{h_1}^2(I_n^{-1}(x)) - z_{h_2}^2(I_n^{-1}(x)), \\ &\quad z_{h_1}^2(I_n^{-1}(x)) - z_{h_2}^2(I_n^{-1}(x))\} \\ &\leq \delta^{12} d_{\mathcal{H}}(h_1, h_2). \end{aligned}$$

where $\delta^{11} = \max_{n=1,2,\dots,K^{11}} \{|\alpha_n^{11}|, |\beta_n^{11}|, |\gamma_n^{11}|\} < 1$ and $\delta^{12} = \max_{n=K^{11}+1,\dots,N} \{|\alpha_n^{12}|, |\beta_n^{12}|, |\gamma_n^{12}|\} < 1$. Therefore

$$d_{\mathcal{F}}(\tilde{f}_1, \tilde{f}_2) \leq \max\{\delta^{11}, \delta^{12}\} \max\{d_{\mathcal{F}}(f_1, f_2), d_{\mathcal{H}}(h_1, h_2)\}.$$

Similarly, one can have

$$d_{\mathcal{H}}(\tilde{h}_1, \tilde{h}_2) \leq \max\{\delta^{21}, \delta^{22}\} \max\{d_{\mathcal{F}}(f_1, f_2), d_{\mathcal{H}}(h_1, h_2)\}.$$

where $\delta^{21} = \max_{n=1,2,\dots,K^{21}} \{|\alpha_n^{21}|, |\beta_n^{21}|, |\gamma_n^{21}|\} < 1$ and $\delta^{22} = \max_{n=K^{21}+1,\dots,M} \{|\alpha_n^{22}|, |\beta_n^{22}|, |\gamma_n^{22}|\} < 1$. Hence

$$d(T(f_1, h_1), T(f_2, h_2)) = \max\{d_{\mathcal{F}}(\tilde{f}_1, \tilde{f}_2), d_{\mathcal{H}}(\tilde{h}_1, \tilde{h}_2)\} \leq \delta \max\{d_{\mathcal{F}}(f_1, f_2), d_{\mathcal{H}}(h_1, h_2)\}$$

where $\delta = \max\{\delta^{11}, \delta^{12}, \delta^{21}, \delta^{22}\} < 1$. Which proves that T is contraction mapping. Then by Banach fixed point theorem, T posses a unique fixed point, say (f_0, h_0) .

Now, for $n = 1, 2, \dots, K^{11}$

$$\begin{aligned} f_0(x_n^1) &= (c_{n+1}^{11} I_{n+1}^{-1}(x_n^1) + \alpha_{n+1}^{11} y_{f_0}^1(I_{n+1}^{-1}(x_n^1)) + \beta_{n+1}^{11} z_{f_0}^1(I_{n+1}^{-1}(x_n^1)) + d_{n+1}^{11}, \\ &\quad \gamma_{n+1}^{11} z_{f_0}^1(I_{n+1}^{-1}(x_n^1)) + e_{n+1}^{11} I_{n+1}^{-1}(x_n^1) + f_{n+1}^{11}) \\ &= (y_n^1, z_n^1). \end{aligned}$$

For $n = K^{11} + 1, \dots, N - 1$

$$\begin{aligned} f_0(x_n^1) &= (c_{n+1-K^{11}}^{12} I_{n+1}^{-1}(x_n^1) + \alpha_{n+1-K^{11}}^{12} y_{h_0}^2(I_{n+1}^{-1}(x_n^1)) + \beta_{n+1-K^{11}}^{12} z_{h_0}^2(I_{n+1}^{-1}(x_n^1)) + d_{n+1-K^{11}}^{12}, \\ &\quad \gamma_{n+1-K^{11}}^{12} z_{h_0}^2(I_{n+1}^{-1}(x_n^1)) + e_{n+1-K^{11}}^{12} I_{n+1}^{-1}(x_n^1) + f_{n+1-K^{11}}^{12}) \\ &= (y_n^1, z_n^1) \end{aligned}$$

This shows that f_0 is the function which interpolates the data $\{(x_n^1, y_n^1, z_n^1) : n = 0, 1, \dots, N\}$. Similarly, it can be shown that g_0 is the function which interpolates the data $\{(x_n^2, y_n^2, z_n^2) : n = 0, 1, \dots, M\}$. Now for $x \in [x_0^1, x_N^1]$ and $x \in [x_0^2, x_M^2]$

$$\begin{aligned} f_0(I_n(x)) &= (c_n^{11} x + \alpha_n^{11} y_{f_0}^1(x) + \beta_n^{11} z_{f_0}^1(x) + d_n^{11}, \\ &\quad \gamma_n^{11} z_{f_0}^1(x) + e_n^{11} x + f_n^{11}) \quad \text{for } n = 1, 2, \dots, K^{11} \end{aligned}$$

$$\begin{aligned} f_0(I_n(x)) &= (c_n^{12} x + \alpha_n^{12} y_{h_0}^2(x) + \beta_n^{12} z_{h_0}^2(x) + d_n^{12}, \\ &\quad \gamma_n^{12} z_{h_0}^2(x) + e_n^{12} x + f_n^{12}) \quad \text{for } n = 1, 2, \dots, K^{12} \end{aligned}$$

and

$$\begin{aligned} h_0(J_n(x)) &= (c_n^{21} x + \alpha_n^{21} y_{f_0}^1(x) + \beta_n^{21} z_{f_0}^1(x) + d_n^{21}, \\ &\quad \gamma_n^{21} z_{f_0}^1(x) + e_n^{21} x + f_n^{21}) \quad \text{for } n = 1, 2, \dots, K^{21} \end{aligned}$$

$$\begin{aligned} h_0(J_n(x)) &= (c_n^{22} x + \alpha_n^{22} y_{h_0}^2(x) + \beta_n^{22} z_{h_0}^2(x) + d_n^{22}, \\ &\quad \gamma_n^{22} z_{h_0}^2(x) + e_n^{22} x + f_n^{22}) \quad \text{for } n = 1, 2, \dots, K^{22}. \end{aligned}$$

If F and H are the graphs of f_0 and h_0 respectively, then

$$\begin{aligned} F &= \bigcup_{i=1}^{K^{11}} w_i^{11}(F) \bigcup \bigcup_{i=1}^{K^{12}} w_i^{12}(H) \\ H &= \bigcup_{i=1}^{K^{21}} w_i^{21}(F) \bigcup \bigcup_{i=1}^{K^{22}} w_i^{22}(H). \end{aligned}$$

Now the uniqueness of the attractor imply that $F = G^1$ and $H = G^2$. That is $G^1 = \{(x, f_0(x)) : x \in I\}$ and $G^2 = \{(x, h_0(x)) : x \in J\}$. \square

Example 3.1. Consider the data sets as

$$D^1 = \{(0, 5), (1, 4), (2, 1), (3, 1), (4, 4), (5, 5)\}$$

$$D^2 = \{(0, 1), (1, 2), (2, 3), (3, 2), (4, 1)\}$$

realizing the graph with $K^{11} = 3$, $K^{12} = 2$, $K^{21} = 1$, $K^{22} = 3$. Take the generalized data set

$$\mathcal{D}^1 = \{(0, 5, 5), (1, 4, 4), (2, 1, 1), (3, 1, 1), (4, 4, 4), (5, 5, 5)\}$$

and

$$\mathcal{D}^2 = \{(0, 1, 1), (1, 2, 2), (2, 3, 3), (3, 2, 2), (4, 1, 1)\}$$

corresponding to D^1 and D^2 respectively. Here $y_n = z_n$ for both the generalized data sets. Choose $\alpha_n^{rs} = 1/3$, $\beta_n^{rs} = 1/3$, $\gamma_n^{rs} = 1/3$ for all $r, s \in \{1, 2\}$ and $n = 1, 2, \dots, K^{rs}$. Then Fig 1 and Fig 2 are the attractors of the corresponding generalized GDIFS.

Keeping the free variables and constrained variables same, Fig 3 and Fig 4 are the attractors of the generalized GDIFS associated with the generalized data sets

$$\mathcal{D}^1 = \{(0, 5, 3), (1, 4, 2), (2, 1, 5), (3, 1, 2), (4, 4, 1), (5, 5, 4)\}$$

$$\mathcal{D}^2 = \{(0, 1, 2), (1, 2, 5), (2, 3, 1), (3, 2, 3), (4, 1, 1)\}.$$

Take the generalized data set

$$\mathcal{D}^1 = \{(0, 5, 3), (1, 4, 2), (2, 1, 5), (3, 1, 2), (4, 4, 1), (5, 5, 4)\}$$

and

$$\mathcal{D}^2 = \{(0, 1, 2), (1, 5, 5), (2, 3, 1), (3, 2, 3), (4, 4, 1)\}$$

corresponding to D^1 and D^2 respectively. Then Fig 5 and Fig 6 are the attractors of the generalized GDIFS with the free variables and constraints variables given in following table 1.

TABLE 1.

| α | α_1^{11} | α_2^{11} | α_3^{11} | α_1^{12} | α_2^{12} | α_1^{21} | α_1^{22} | α_2^{22} | α_3^{22} |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| | 0.8 | 0.7 | 0.8 | 0.7 | 0.8 | 0.99 | 0.99 | 0.99 | 0.99 |
| β | β_1^{11} | β_2^{11} | β_3^{11} | β_1^{12} | β_2^{12} | β_1^{21} | β_1^{22} | β_2^{22} | β_3^{22} |
| | -0.3 | -0.4 | -0.2 | -0.3 | -0.4 | 0.99 | 0.99 | 0.99 | 0.99 |
| γ | γ_1^{11} | γ_2^{11} | γ_3^{11} | γ_1^{12} | γ_2^{12} | γ_1^{21} | γ_1^{22} | γ_2^{22} | γ_3^{22} |
| | 0.5 | 0.3 | 0.6 | 0.5 | 0.3 | 0.005 | 0.005 | 0.005 | 0.005 |

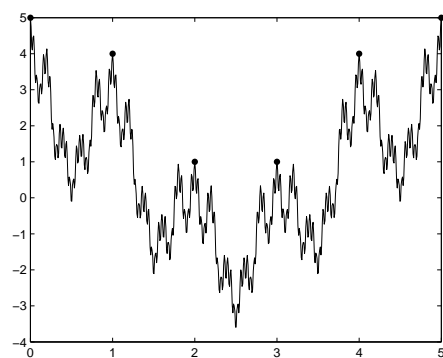


FIGURE 1.

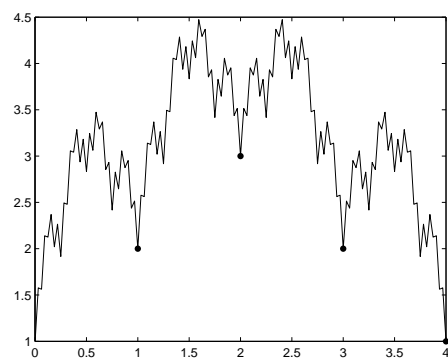


FIGURE 2.

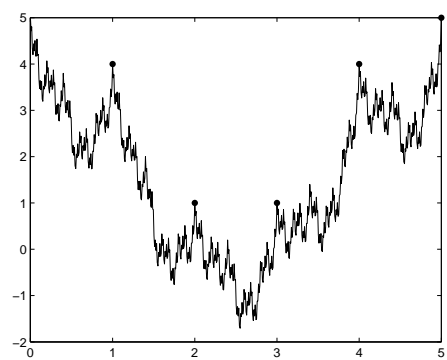


FIGURE 3.

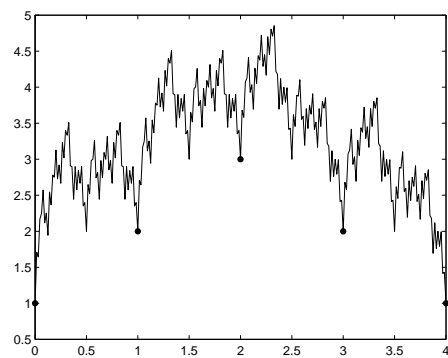


FIGURE 4.

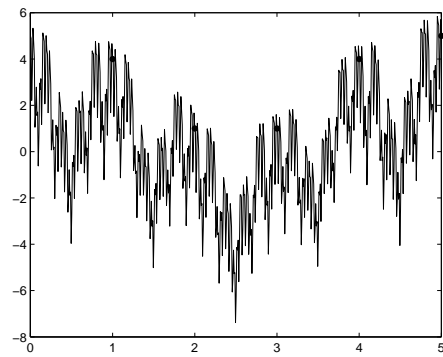


FIGURE 5.

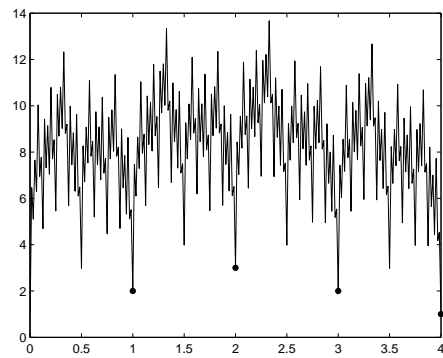


FIGURE 6.